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NUMERICAL METHODS

Completely Conservative Difference Schemes for Dynamic Problems of Linear Elasticity and Viscoelasticity

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Abstract—On the basis of a mixed statement (velocity–strain), we complete the development of a general theory of completely conservative adjoint-coordinated difference schemes for dynamic problems of linear elasticity and viscoelasticity. In particular, our explicitly solvable discrete models permit controlling the total energy imbalance and have the same parallelization degree as the conventional explicit schemes.

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Let V be a connected domain where the Cartesian coordinates of a point $M(x_1, x_2, x_3) \in V$ vary, and let the boundary $\gamma = \gamma_1 \cup \gamma_2$ of V be "sufficiently smooth." If t is time, then $G = [0 \le t \le t_1] \times V$ is the phase volume. The parameters of the dynamic process in question are as follows: $u(M,t)$ is the displacement vector, $v(M,t) = \partial u/\partial t$ is the velocity, $\varepsilon(M,t)$ is the strain tensor, and $\sigma(M,t)$ is the stress tensor. Mathematical models of dynamic problems of linear elasticity are usually based on the momentum conservation law

$$\varrho \frac{\partial^2 u}{\partial t^2} + R^* \sigma = \varrho f \leftrightarrow \varrho \frac{\partial v}{\partial t} + R^* \sigma = \varrho f, \tag{1}$$

the constitutive "displacement–strain" relation

$$\varepsilon = Ru \leftrightarrow 2\varepsilon_{ij} = \partial_j u_i + \partial_i u_j, \qquad \partial_i = \partial/\partial x_i, \qquad \varepsilon_{ij} = \varepsilon_{ji}, \tag{2}$$

and the equation of state

$$\sigma = K\varepsilon \leftrightarrow \sigma_{ij} = 2\mu\varepsilon_{ij} + \lambda\delta_{ij}(\varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}) = \sigma_{ji}. \tag{3}$$

In (1)–(3), the vectors $u = (u_1, u_2, u_3)^{\mathrm{T}}$ and $v = (v_1, v_2, v_3)^{\mathrm{T}}$ are treated as elements of a Hilbert space H, and the symmetric tensors

$$\varepsilon = (\varepsilon_{11}, \varepsilon_{22}, \varepsilon_{33}, 2\varepsilon_{12}, 2\varepsilon_{13}, 2\varepsilon_{23})^{\mathrm{T}}, \qquad \sigma = (\sigma_{11}, \sigma_{22}, \sigma_{33}, \sigma_{12}, \sigma_{13}, \sigma_{23})^{\mathrm{T}}$$

of rank 2 are treated as elements of the Hilbert space H^*. The inner products on H and H^* are defined as the convolution of the corresponding elements. The vector components u_i and v_i and the tensor entries ε_{ij} and σ_{ij} are scalar functions of the vector argument $(M,t) \in G$. For given scalar functions $\varrho(M) > 0$, $\lambda(M) > 0$, $\mu(M) > 0$, and $f(M)$, system (1)–(3) is closed in the sense that the number of independent relations (1)–(3) exactly coincides with the number of parameters $u_i(M,t)$, $v_i(M,t)$, $\varepsilon_{ij}(M,t)$, and $\sigma_{ij}(M,t)$ to be determined.

System (1)–(3) should be supplemented with the initial conditions

$$u(M,0) = \varphi_1(M), \qquad v(M,0) = \varphi_2(M) \tag{4}$$

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and boundary conditions, say,

$$u|_{\gamma_1} = 0, \qquad \sigma_{ij} n_j|_{\gamma_2} = 0, \tag{5}$$

where the n_j are the components of the unit outward normal on γ_2. The particular boundary conditions for (1)–(3) are always determined by the specific features of the problem in question. The choice in (5) (the boundary γ_1 is fixed, and the boundary γ_2 is stress-free) has only been made to simplify the subsequent exposition as much as possible. This also pertains to the definition of the function $f(M)$ in (1). For example, if $\gamma_1 = 0$ in (5), then the balance conditions do not permit one to specify the function $f(M)$ arbitrarily.

By (2), the operator $R:\ H \to H^*$ is defined as follows:

$$\varepsilon = \begin{pmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{12} \\ 2\varepsilon_{13} \\ 2\varepsilon_{23} \end{pmatrix} = \begin{pmatrix} \partial_1 & 0 & 0 \\ 0 & \partial_2 & 0 \\ 0 & 0 & \partial_3 \\ \partial_2 & \partial_1 & 0 \\ \partial_3 & 0 & \partial_1 \\ 0 & \partial_3 & \partial_2 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = Ru. \tag{6}$$

By definition, for the self-adjoint operator $R^*:\ H^* \to H$ from (1), we have

$$(u, R^*\sigma)_H = (Ru, \sigma)_{H^*}, \qquad u \in H, \qquad \sigma \in H^*. \tag{7}$$

This, together with relations (5) and (6) and the integration by parts formula, implies that

$$R^*\sigma = -R^{\mathrm{T}}\sigma = -\begin{pmatrix} \partial_1 & 0 & 0 & \partial_2 & \partial_3 & 0 \\ 0 & \partial_2 & 0 & \partial_1 & 0 & \partial_3 \\ 0 & 0 & \partial_3 & 0 & \partial_1 & \partial_2 \end{pmatrix} \begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{13} \\ \sigma_{23} \end{pmatrix} = -\operatorname{div}\sigma, \tag{8}$$

which, in the case of a Cartesian reference system, coincides with the definition of the divergence of a tensor of rank 2. Finally, by (3),

$$K = \begin{pmatrix} K_1 & 0 \\ 0 & K_2 \end{pmatrix}, \qquad K_1 = \begin{pmatrix} \lambda + 2\mu & \lambda & \lambda \\ \lambda & \lambda + 2\mu & \lambda \\ \lambda & \lambda & \lambda + 2\mu \end{pmatrix}, \qquad K_2 = \begin{pmatrix} \mu & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \mu \end{pmatrix}. \tag{9}$$

Therefore, if $\lambda(M) > 0$ and $\mu(M) > 0$ in (3), then $K = K^{\mathrm{T}} > 0$.

Therefore, we arrive at the following closed operator mathematical model for dynamic problems of linear elasticity:

$$\begin{aligned} &\varrho\frac{\partial v}{\partial t} + R^*\sigma = \varrho f, \qquad \sigma = K\varepsilon, \qquad \varepsilon = Ru, \qquad \frac{\partial u}{\partial t} = v, \\ &u(M,0) = \varphi_1(M), \qquad \frac{\partial u}{\partial t}(M,0) = v(M,0) = \varphi_2(M), \qquad (M,t) \in G. \end{aligned} \tag{10}$$

The boundary conditions (5) are used in (10) to specify the domains of the operators R and R^*. When carrying out a numerical experiment for the considered class of dynamic problems, one should consider the model (10) as a component of the famous Samarskii triad: I model → II algorithm → III computer software. Therefore, roughly speaking, the "performance" of a continuous $[(M,t) \in G]$

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model at stage I often determines the "performance" of the discrete $[(M_h, n\tau) \in G_h]$ model at the same stage and hence the "performance" of the numerical experiment for the considered class of problems. Let us explain this using the continuous model (10) as an example.

First, note that relation (10) does not a priori prescribe the order in which the unknown parameters $u(M,t)$, $v(M,t)$, $\varepsilon(M,t)$, and $\sigma(M,t)$ should be found. At the same time, as a basis of numerical experiment for the considered class of dynamic problems, one uses continuous models with prescribed order in which the unknown parameters should be found. For example, consider the statement of dynamic problems in "stresses,"

$$\sigma = K\varepsilon \leftarrow \varepsilon \rightarrow u \rightarrow v = \partial u/\partial t, \tag{11}$$

or in "displacements,"

$$v = \partial u/\partial t \leftarrow u \rightarrow \varepsilon \rightarrow \sigma = K\varepsilon. \tag{12}$$

In (11), the parameter $\varepsilon \in H^*$ is determined first, and in (12), one first finds the parameter $u \in H$.

In connection with the passage $\varepsilon \rightarrow u$ in (11), consider the operator equation

$$Ru = \varepsilon. \tag{13}$$

Lemma 1. *The operator equation* (3) *is solvable if and only if*

$$(\varepsilon, \psi)_{H^*} = 0, \qquad R^*\psi = 0. \tag{14}$$

One can show that the Saint-Venant compatibility (solidity) condition, vanishing of the incompatibility tensor, and condition (14) are equivalent. If condition (14) is satisfied, then the element $u \in H$ in Eq. (13) is determined by the element $\varepsilon \in H^*$ modulo an element of the kernel of the operator R,

$$u = R^{-1}\varepsilon + \widehat{u}, \qquad R\widehat{u} = 0. \tag{15}$$

The element $\widehat{u}$ (the rigid displacement vector) is usually fixed with the use of a boundary condition on γ_1, and then the passage $\varepsilon \rightarrow u$ can be performed with the use of the Cauchy–Cesaro condition. Condition (14) determines the solvability subspace $H_1^* \subset H^*$ of problem (13) in H^*. Therefore, the statement (11) of the dynamic problem in "stresses" should be supplemented with the condition $\varepsilon \in H_1^*$.

Now consider the statement (11) in "displacements." Here, for the initial determination of the "displacements," the momentum conservation law (1) is transformed as follows:

$$\varrho\frac{\partial^2 u}{\partial t^2} + R^*\sigma = \varrho f \rightarrow \varrho\frac{\partial^2 u}{\partial t^2} + R^*K\varepsilon = \varrho f \rightarrow \varrho\frac{\partial^2 u}{\partial t^2} + R^*KRu = \varrho f. \tag{16}$$

But the last passage in (16) is related to the *assumption* that, to each $\varepsilon \in H^*$, there corresponds an element $u \in H$ such that $\varepsilon = Ru$. This means that $\varepsilon \in H_1^*$ in (16).

Lemma 2. *If*

$$\frac{\partial \varepsilon}{\partial t} - Rv = 0, \qquad \varepsilon(M,0) = Ru(M,0) = R\varphi_1(M), \tag{17}$$

then the solvability condition (14) *is satisfied for* $0 \le t \le t_1$, *which implies that* $\varepsilon(M,t) \in H_1^*$.

Lemma 3. *Problem* (17) *and the problem*

$$\frac{\partial u}{\partial t} = v, \qquad u(M,0) = \varphi_1(M) \tag{18}$$

are equivalent.

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This is sufficient to justify the following choice of a continuous mathematical model (stage I of the triad I → II → III) for dynamic problems of linear elasticity:

$$\begin{aligned} &\varrho\frac{\partial v}{\partial t}+R^*\sigma=\varrho f, \qquad \frac{\partial \varepsilon}{\partial t}-Rv=0, \qquad \frac{\partial u}{\partial t}=v, \qquad \sigma=K\varepsilon,\\ &u(M,0)=\varphi_1(M), \qquad v(M,0)=\varphi_2(M), \qquad \varepsilon(M,0)=R\varphi_1(M). \end{aligned} \tag{19}$$

In the model (19), the boundary conditions (5) are used [as well as in the model (10)] to specify the domains of the operators R and R^*.

Theorem 1. *The additional conservation law*

$$\frac{\partial I(t)}{\partial t}=(\varrho f,v)_H, \qquad I(t)=0.5(\varrho v,v)_H+0.5(\varepsilon,\sigma)_{H^*}, \qquad \sigma=K\varepsilon, \tag{20}$$

holds for the continuous model (19).

Proof. The desired assertion is almost obvious. One should take the inner products of the first relation in (19) by v and the second relation by σ and add the results.

The physical interpretation of the conservation law (20) is equally obvious. By integrating (20) with respect to t and by using the relation $v=\partial u/\partial t$, we obtain

$$I(t+\Delta t)=I(t)+\int\limits_t^{t+\Delta t}\left(\varrho f,\frac{\partial u}{\partial t}\right)_H dt=I(t)+Q_1(t+\Delta t). \tag{21}$$

Consider relation (20) and note that the first term in $I(t)$ corresponds to the kinetic energy of the elastic medium and the second term to the potential elastic strain energy, both related to the volume V. Finally, Q_1 in relation (21) is the work of mass forces on the displacement increments Δu in time Δt. Therefore, relation (20) or (21) can be treated as the conservation law for the total energy of the elastic medium in the volume V.

The basic model (19) associated with the original definition of the covector $(v,\varepsilon)^{\mathrm{T}}$ can also be used as a basis of a numerical experiment for problems of dynamics of a linear viscoelastic medium. Consider the Kelvin (Kelvin–Voigt) viscoelastic medium. Let

$$\begin{aligned} \varepsilon&=(\varepsilon_{11},\varepsilon_{22},\varepsilon_{33},0,0,0)^{\mathrm{T}}+(0,0,0,2\varepsilon_{12},2\varepsilon_{13},2\varepsilon_{23})^{\mathrm{T}}=(\varepsilon_1,0)^{\mathrm{T}}+(0,\varepsilon_2)^{\mathrm{T}}=(\varepsilon_1,\varepsilon_2)^{\mathrm{T}},\\ \sigma&=(\sigma_{11},\sigma_{22},\sigma_{33},0,0,0)^{\mathrm{T}}+(0,0,0,\sigma_{12},\sigma_{13},\sigma_{23})^{\mathrm{T}}=(\sigma_1,0)^{\mathrm{T}}+(0,\sigma_2)^{\mathrm{T}}=(\sigma_1,\sigma_2)^{\mathrm{T}}. \end{aligned}$$

Such representations correspond to the expansion of the strain tensor and the stress tensor into the spherical and deviatory parts. In view of the notation in (19), the state equation $\sigma=K\varepsilon$ for a linear elastic medium can be represented in the form

$$\sigma_1=K_1\varepsilon_1, \qquad \sigma_2=K_2\varepsilon_2. \tag{22}$$

Unlike (22), the state equations for the Kelvin viscoelastic medium are defined as follows:

$$\widehat{\sigma}_1=K_1\widehat{\varepsilon}+\eta_1\frac{\partial\widehat{\varepsilon}_1}{\partial t}, \qquad \widehat{\sigma}_2=K_2\widehat{\varepsilon}+\eta_2\frac{\partial\widehat{\varepsilon}_2}{\partial t}, \tag{23}$$

where scalar functions $\eta_1,\eta_2>0$ of a vector argument specify the bulk and shear viscosities of the medium. For simplicity, in the following, we set $\eta_1=\eta_2=\eta$, and then

$$\widehat{\sigma}=K\widehat{\varepsilon}+\eta\frac{\partial\widehat{\varepsilon}}{\partial t}. \tag{24}$$

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Thus, for problems of dynamics of a Kelvin viscoelastic medium, we obtain the closed operator model

$$\begin{aligned} &\varrho\frac{\partial\widehat{v}}{\partial t}+R^*\widehat{\sigma}=\varrho f, \qquad \frac{\partial\widehat{\varepsilon}}{\partial t}-R\widehat{v}=0, \qquad \frac{\partial\widehat{u}}{\partial t}=\widehat{v}, \qquad \widehat{\sigma}=K\widehat{\varepsilon}+\eta R\widehat{v},\\ &\widehat{u}(M,0)=\varphi_1(M), \qquad \widehat{v}(M,0)=\varphi_2(M), \qquad \widehat{\varepsilon}(M,0)=R\varphi_1(M). \end{aligned} \tag{25}$$

The boundary conditions for (25) are defined in (5), where σ should be replaced by $\widehat{\sigma}$ and u by $\widehat{u}$. In (25), as well as in (19), the covector $(\widehat{v},\widehat{\varepsilon})^{\mathrm{T}}$ should be determined first.

Theorem 2. *The additional conservation law*

$$\frac{\partial\widehat{I}}{\partial t}+(\widehat{v},R^*\eta R\widehat{v})_H=(\varrho f,\widehat{v})_H \tag{26}$$

holds for the continuous model (25). *In* (26), η *should be treated as the matrix* K *in* (9) *with entries* $k_{ij}=\delta_{ij}\eta$. *For* $\widehat{I}(t)$, *from* (26), *one obtains* [*cf.* (20)]

$$\widehat{I}(t)=0.5(\varrho\widehat{v},\widehat{v})_H+0.5(K\widehat{\varepsilon},\widehat{\varepsilon})_{H^*}, \qquad K\widehat{\varepsilon}\neq\widehat{\sigma}. \tag{27}$$

Remark 1. Continuous models with an additional conservation law are referred to as "entropy" models (S.K. Godunov). Discrete models (difference schemes) with this property are said to be "completely conservative" (Popov–Samarskii [1; 2, Chap. III, Secs. 5, 6]). We mainly pay attention to the construction of completely conservative discrete models and efficient algorithms of their implementation for problem (25). For problem (19), these issues are presented in [3] in sufficient detail. Part of the results in [3] is given here without additional references and explanations.

The passage from continuous to discrete models is carried out in accordance with [4]. Notation that is standard in the theory of difference schemes will be used without explanation. Thus,

$$\begin{aligned} &G\to G_h=[0,\tau,\dots,n\tau,\dots,k\tau=t_1]\times V_h, \qquad M_h\in V_h, \qquad w_h^n=w(M_h,n\tau),\\ &\gamma\to\gamma_h=\gamma_{1h}\cup\gamma_{2h}, \qquad u|_{\gamma_1}=0\to u_h^n|_{\gamma_{1h}}=0, \qquad \sigma_{ij}n_j|_{\gamma_2}=0\to(\sigma_{ij}^n)_hn_j|_{\gamma_{2h}}=0,\\ &\tau(\cdot)_{\bar t}^{n+1}=(\cdot)_h^{n+1}-(\cdot)_h^n, \qquad \partial_j^+(\cdot)_h=(\cdot)_{x_j}, \qquad \partial_j^-(\cdot)_h=(\cdot)_{\bar x_j}, \qquad (\cdot)_\alpha=\alpha(\cdot)_h^{n+1}+(1-\alpha)(\cdot)_h^n. \end{aligned}$$

The grid $M_h\in V_h$ with respect to the space variables x_j is assumed to be uniform and coordinated with γ_h. The adjoint-coordinated approximation condition [5]

$$(Ru,\sigma)_{H^*}=(u,R^*\sigma)_H\to(R_hu_h,\sigma_h)_{H_h^*}=(u_h,R_h^*\sigma_h)_{H_h} \tag{28}$$

is a key condition in this passage. Condition (28) permits completely preserving the structure of the continuous models (19) and (25) on the discrete level and possibly inheriting some of their other properties.

To the continuous model (19), we assign the two-parameter ($0\le\alpha,\ \beta\le1$) family of adjoint-coordinated two-layer difference schemes

$$\begin{aligned} &\varrho_hv_{\bar t}^{n+1}+R_h^*\sigma_\alpha=\varrho_hf_h, \qquad \varepsilon_{\bar t}^{n+1}-R_hv_\beta=0, \qquad u_{\bar t}^{n+1}=v_\beta, \qquad \sigma_h=K_h\varepsilon_h,\\ &u_h^0=\varphi_{1h}, \qquad v_h^0=\varphi_{2h}, \qquad \varepsilon_h^0=R_h\varphi_{1h}. \end{aligned} \tag{29}$$

By virtue of condition (28), for particular values of α and β, the difference scheme (29) is defined to within the choice of the approximation R_h to the operator R or R_h^* to the operator R^*. In turn, this necessitates coordinating the approximations to the operator ∂_j in (6) and (8). For example, if the approximation ∂_j^- is chosen for ∂_j in (6), then one should take ∂_j^+ for the approximation to ∂_j in (8). One can treat ϱ_h, K_h, f_h, φ_{1h}, and φ_{2h} in (29) as the projection of a given object onto V. Part of the indices can sometimes be omitted to simplify the notation. Next, let

$$I_h^n=0.5(\varrho_hv_h^n,v_h^n)_{H_h}+0.5(\varepsilon_h^n,\sigma_h^n)_{H_h^*}. \tag{30}$$

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Theorem 3. *If $\alpha = \beta = 0.5$, then the difference scheme* (29) *has the same conservation law as the continuous model* (19),

$$I_h^{n+1} = I_h^n + \tau(\varrho_h f_h, v_{0.5})_{H_h}; \tag{31}$$

i.e., the scheme (29) *is completely conservative.*

In a similar way, to the continuous viscoelastic model (25), we assign the family of adjoint-coordinated discrete models

$$\begin{gathered}\varrho_h \widehat{v}_{\bar{t}}^{\,n+1} + R_h^* \widehat{\sigma}_\alpha = \varrho_h f_h, \qquad \widehat{\varepsilon}_{\bar{t}}^{\,n+1} = R_h \widehat{v}_\beta, \quad \widehat{u}_{\bar{t}}^{\,n+1} = \widehat{v}_\beta, \qquad \widehat{\sigma}_h = K_h \widehat{\varepsilon}_h + \eta_h R_h \widehat{v}_\beta, \\ \widehat{u}_h^0 = \varphi_{1h}, \qquad \widehat{v}_h^0 = \varphi_{2h}, \qquad \widehat{\varepsilon}_h^{\,0} = R_h \varphi_{1h}.\end{gathered} \tag{32}$$

Now let

$$\widehat{I}_h^n = 0.5(\varrho_h \widehat{v}_h^n, \widehat{v}_h^n)_{H_h} + 0.5(\widehat{\varepsilon}_h^{\,n}, K_h \widehat{\varepsilon}_h^{\,n})_{H_h^*}. \tag{33}$$

Theorem 4. *If $\alpha = \beta = 0.5$, then the difference scheme* (32) *has the same conservation law as the continuous model* (25),

$$\widehat{I}_h^{n+1} + \tau(\widehat{v}_{0.5}, R_h^* \eta_h R_h \widehat{v}_{0.5})_{H_h} = \widehat{I}_h^n + \tau(\varrho_h f_h, \widehat{v}_{0.5})_{H_h}; \tag{34}$$

i.e., the scheme (32) *is completely conservative.*

The presence of conservation laws (31) and (34) permits one to study the convergence of the difference schemes (29) and (32) readily. Relations (31) and (34) obviously imply uniform stability with respect to the initial data in the energy norms generated by the functionals (30) and (31). Stability with respect to the right-hand side can be obtained in a standard way with the use of the ε-inequality [6, Chap. II, Sec. 2]. However, the derivation of convergence theorems for the difference schemes (29) and (32) is only easy for $\alpha = \beta = 0.5$. A violation of this condition results in some difficulties, part of which are illustrated below for the difference scheme (29). When doing so, it is harmless to set $f_h = 0$ in (29) as well as in forthcoming considerations.

Lemma 4. *The difference scheme* (19) *has the additional "conservation law"*

$$\begin{gathered}I_h^{n+1} + \tau^2 Q(\alpha, \beta) = I_h^n, \qquad Q(\alpha, \beta) = -0.5[(1 - 2\beta)Q(\sigma_\alpha) + (1 - 2\alpha)Q(v_\beta)], \\ Q(\sigma_\alpha) = (R_h \varrho^{-1} R_h^* \sigma_\alpha, \sigma_\alpha)_{H_h^*}, \qquad Q(v_\beta) = (R_h^* K R_h v_\beta, v_\beta)_{H_h}.\end{gathered} \tag{35}$$

For the difference scheme (19), the term $\tau^2 Q(\alpha, \beta)$ in (35) characterizes the total energy imbalance under the passage $I_h^n \to I_h^{n+1}$. It would be more meaningful to have an imbalance characteristic for the total computation time $t_1 = k\tau$ under the passage $I_h^0 \to I_h^k$. To obtain the corresponding estimates, we use the following assertion proved in [7].

Lemma 5. *The nonzero eigenvalues ν_m and δ_m of the spectral problems*

$$R_h \varrho^{-1} R_h^* \eta_m = \nu_m K^{-1} \eta_m, \qquad \eta_m \in H, \qquad R_h^* K R_h r_m = \delta_m \varrho r_m, \qquad r_m \in H_h, \tag{36}$$

coincide, and one has the inequalities

$$\begin{gathered}\nu_{\min}(\varepsilon_\alpha, \sigma_\alpha)_{H_h^*} \le Q(\sigma_\alpha) \le \nu_{\max}(\varepsilon_\alpha, \sigma_\alpha)_{H_h^*}, \\ \nu_{\min}(\varrho v_\beta, v_\beta)_{H_h} \le Q(v_\beta) \le \nu_{\max}(\varrho v_\beta, v_\beta)_{H_h}.\end{gathered} \tag{37}$$

As a rule, the occurrence of the term $\tau^2 Q(\alpha, \beta)$ in the "conservation law" (35) is attributed to the "approximation viscosity" of the difference scheme (19), where no physical meaning is usually ascribed to the word "viscosity." But then one ask what the *physical* process is to which the discrete model (19) with the "conservation law" (35) corresponds. For $\alpha = \beta = 0.5$, the answer is given by Theorem 3.

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Remark 2. The physical processes described by the continuous models (19) and (25) are essentially different. We deal with an equilibrium (invertible) process in the first case and a noninvertible process in the second case. The entropy $S(t)$ related to a given volume V characterizes the noninvertibility of the process. For the viscoelastic model (25), we have [8, Part 1, Sec. 16]

$$\varrho T\frac{dS}{dt}=\Phi,\qquad \Phi=(R^*\eta Rv,v)_H, \tag{$*$}$$

where T is temperature and Φ is a dissipative function characterizing the change in the mechanical energy $I(t)$ under the transition $t\to t+\Delta t$.

Next, note that, by virtue of (35), the properties of the discrete elastic model (29) substantially depend on the sign of $Q(\alpha,\beta)$. Namely, if S is the entropy, then

$$Q(\alpha,\beta)>0\to I_h^{n+1}<I_h^n\to S^{n+1}>S^n, \tag{38_1}$$
$$Q(\alpha=\beta=0.5)=0\to I_h^{n+1}=I_h^n\to S^{n+1}=S^n, \tag{38_2}$$
$$Q(\alpha,\beta)<0\to I_h^{n+1}>I_h^n\to S^{n+1}<S^n. \tag{38_3}$$

Apparently, the use of the discrete model in (38_3) is hardly expedient for a numerical experiment, at the very least, because $dS=0$ for the original continuous discrete model (19). Nevertheless, we illustrate some properties of the discrete model (38_3) using explicit difference scheme (29), $\alpha=\beta=0$, as an example. In the explicit difference scheme, I_h^{n+1} increases infinitely, and this growth should be restricted to obtain an estimate for the stability of the difference scheme with respect to the initial data. We have

$$I_h^k=I_h^{k-1}+0.5\tau^2[Q(\sigma^{k-1})+Q(v^{k-1})]\ge(1+\tau^2\nu_{\max})I_h^{k-1};$$

consequently,

$$I_h^k\le(1+\tau^2\nu_{\max})^kI_h^0. \tag{39}$$

There is a well-known estimate for $\nu_{\max}$, $\nu_{\max}\le c^2/h^2$, where the domain constant c^2 is independent of h. Let us specify the passage to the limit as follows: $\tau^2/h^2=\delta^2=\text{const}$ as $\tau\to0$ and $h\to0$. In addition to (39), we obtain the following stability estimate with respect to the initial data:

$$I_h^k\le MI_h^0,\qquad M=\exp(a^2t_1),\qquad a^2=c^2\delta^2. \tag{40}$$

Along with (39), we have a similar upper bound for the mechanical energy imbalance after the termination of computations: $t_1=k\tau$, and so

$$(1+\tau\delta^2h^2\nu_{\max})^{-k}\le I_h^0/I_h^k\le(1+\tau\delta^2h^2\nu_{\min})^{-k}. \tag{41}$$

From the formal viewpoint, the estimate (41) permits one to use τ and δ^2 to "control the mechanical energy imbalance" for the discrete model (38_3) with $\alpha=\beta=0$. However, here one should take into account the fact that, when using inequalities (37), the equality in the upper and lower bounds is attained at distinct elements of the spectral problems (36). In addition, since $S_{\bar t}^{n+1}<0$, we see that the model (38_3) with $\alpha=0$ and $\beta=0$ itself can only be treated as a hypothetical one and only as an abstract approximation to the discrete model (38_2). Therefore, the computation of I_h^0 and I_h^k is absolutely necessary when analyzing real problems with the use of the model (38_3) with $\alpha=\beta=0$.

The same functionals should be computed when using the discrete model (38_1) with $\alpha=\beta=1$ (an implicit difference scheme) for the computations. Here $S_{\bar t}^{n+1}>0$ and

$$I_h^{n+1}\le I_h^n\le\cdots\le I_h^0, \tag{42}$$

which implies that one has uniform stability with respect to the initial data and sufficient conditions for the convergence are satisfied. However, one immediately needs to find out whether I_h^n in (42)

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is bounded below. In this connection, consider the discrete *elastic* model (38_1) with $\alpha = 1$ and $\beta = 0.5$. For this model, we have

$$I_h^{n+1} + \tau^2(R_h^* K_h R_h \widehat{v}_{0.5}, \widehat{v}_{0.5})_{H_h} = I_h^n. \tag{43}$$

At the same time, for the discrete *viscoelastic* model (32) with $\alpha = \beta = 0.5$, we have [see (34)]

$$\widehat{I}_h^{n+1} + \tau^2(R_h^* \eta_h R_h \widehat{v}_{0.5}, \widehat{v}_{0.5})_{H_h} = \widehat{I}_h^n. \tag{44}$$

The conservation laws (43) and (44) coincide to within notation if we set $\eta_h = K_h\tau^2$ in the latter. This coincidence is neither formal nor random. Indeed, on the one hand, the vector $\widehat{w}(M,t) = (\widehat{v}, \widehat{u}, \widehat{\varepsilon}, \widehat{\sigma})^{\mathrm{T}}$ in the viscoelastic model (25) satisfies the relation [9]

$$\widehat{w}(M,t) \to w(M) = (0,\ Lu = \varrho f,\ \varepsilon = Ru,\ \sigma = K\varepsilon)^{\mathrm{T}}, \qquad L = R^* K R, \tag{45}$$

as $t \to \infty$, where L is the Lamé operator, which implies exit to a stationary mode. On the other hand, if the asymptotic stability condition [6, Chap. III, Secs. 2, 3] is satisfied, then the exit to a stationary mode occurs for the discrete elastic model (38_1) with $\alpha = 1$ and $\beta = 0.5$ as well. Namely, if $k \to \infty$, then

$$w_h^k(M, n\tau) \to w_h(M_h) = (0,\ L_h u_h = \varrho_h f_h,\ \varepsilon_h = R_h u_h,\ \sigma_h = K_h \varepsilon_h). \tag{46}$$

The comparison of relations (45) with (46) shows that there exist asymptotically stable discrete elastic models (38_1) for which there onsets a "regular" stationary mode as $\tau \to 0$, $h \to 0$, and $t \to \infty$; which mode corresponds to the static linear elasticity problem

$$R^* K R u \equiv Lu = \varrho f, \qquad \varepsilon = Ru, \qquad \sigma = K\varepsilon, \qquad u|_{\gamma_1} = 0, \qquad \sigma_{ij} n_j|_{\gamma_2} = 0. \tag{47}$$

For us, what was said above is more than enough reason to choose the discrete viscoelastic model (32) with $\alpha = \beta = 0.5$ as a basis for the construction of efficient algorithms under the passage $n \to n+1$. First, we rewrite the scheme (32) with $\alpha = \beta = 0.5$ in the canonical form ($f = 0$)

$$\begin{gathered} By_{\bar{t}}^{n+1} + \widehat{A} y^n = 0, \quad B = 0.5\tau\widehat{A}, \quad \widehat{A} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & 0 \end{pmatrix}, \quad y^n = w(M_h, n\tau) = (v_h^n, \varepsilon_h^n)^{\mathrm{T}}, \\ A_{11} = \varrho^{-1} R_h^* \eta R_h = \varrho^{-1} L_h(\eta), \qquad A_{12} = \varrho^{-1} R_h K_h, \qquad A_{21} = -R_h. \end{gathered} \tag{48}$$

The canonical form (48) takes into account the decomposition

$$\widehat{\sigma} = \sigma + \sigma' = K\varepsilon + \eta \frac{\partial \varepsilon}{\partial t} = K\varepsilon + \eta R v$$

of the tensor $\widehat{\sigma}$ in (32) into the sum of the elastic stress tensor σ and the viscous stress tensor σ'. The initial data and the boundary conditions in (48) are the same as in (32).

The passage $n \to n+1$ in (48) requires the inversion of the operator $B = (E + 0.5\tau\widehat{A})$. The class of economical implementations (algorithms) considered below uses special additive decompositions of the operator $\widehat{A}$ and the construction of the corresponding factorization of the operator $\widehat{B}$,

$$\widehat{A} = \sum_{i=1}^m \widehat{A}^{(i)} \to \widehat{B} = \prod_{i=1}^m \widehat{B}^{(i)} = \prod_{i=1}^m (E + 0.5\tau\widehat{A}^{(i)}).$$

Then, instead of the discrete model (48), one uses the discrete model

$$B^{(1)} \cdots B^{(m)} y_{\bar{t}}^{n+1} + \widehat{A} y^n = 0, \tag{49}$$

and, say, for $m = 2$, the passage $n \to n+1$ in (49) can be carried out in a standard way,

$$B^{(1)} y^* + \widehat{A} y^n = 0, \qquad B^{(2)} y_{\bar{t}}^{n+1} = y^*. \tag{50}$$

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Remark 3. The idea itself of the passage from the operator B in (48) to the factorized operator $\widehat{B}$ in (49) is usually treated as an "approximate factorization method." An algorithmic implementation of this idea, depending on the specific problem, is treated as either the "method of decomposition in the space variables," or a "locally one-dimensional method," or, finally, the "method of decomposition into physical processes" [10, Chap. VI, Sec. 30]. As to the last method, $\varrho A_{11} = L_h(\eta) \to 0$ in (48) as $\eta \to 0$, and it is most natural to use the additive decomposition of the operator $\widehat{A}$ into "physical processes,"

$$\widehat{A} = \begin{pmatrix} 0 & A_{12} \\ A_{21} & 0 \end{pmatrix} + \begin{pmatrix} A_{11} & 0 \\ 0 & 0 \end{pmatrix} = \widehat{A}^{(1)} + \widehat{A}^{(2)}. \tag{51}$$

Then the discrete model for finding y^* in (50) coincides, to within notation, with the discrete elastic model (29) with $\alpha = \beta = 0.5$, which has the conservation law (31). Therefore, by (51), the problem of constructing economical implementations (algorithms) for the discrete viscoelastic model (32) with $\alpha = \beta = 0.5$ is directly related to the problem of constructing an economical algorithm for the discrete elastic model (29) with $\alpha = \beta = 0.5$. The latter problem was studied rather comprehensively in [3].

Let us present a result in [3] to be used in what follows. In the canonical representation of the discrete operator model (29) with $\alpha = \beta = 0.5$, $y = w_h$,

$$(E+0.5\tau A)y_{\bar{t}}^{n+1}+Ay^n = 0, \qquad A = \begin{pmatrix} 0 & A_{12} \\ A_{21} & 0 \end{pmatrix} = \begin{pmatrix} 0 & A_{12} \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ A_{21} & 0 \end{pmatrix} = A^{(1)}+A^{(2)}, \tag{52}$$

we replace the operator $B = (E + 0.5\tau A)$ by the factorized operator

$$\widetilde{B} = B_1B_2 = \begin{pmatrix} E & 0.5\tau A_{12} \\ 0 & E \end{pmatrix}\begin{pmatrix} E & 0 \\ 0.5\tau A_{21} & E \end{pmatrix}, \qquad \widetilde{B} = B + O(\tau^2).$$

This corresponds to the additive expansion of the operator A indicated in (52) with the subsequent approximate factorization (49) and the standard implementation (50). The standard implementation leads to the problem

$$v^* + 0.5\tau A_{12}\varepsilon^* + A_{12}\varepsilon^n = 0, \qquad \varepsilon^* + A_{21}v^n = 0, \tag{50_1}$$

for the passage $y^n \to y^*$ and to the problem

$$v_{\bar{t}}^{n+1} = v^*, \qquad 0.5\tau A_{21}v_{\bar{t}}^{n+1} + \varepsilon_{\bar{t}}^{n+1} = \varepsilon^* \tag{50_2}$$

for the passage $y^* \to y^{n+1}$. Obviously, it follows from (50_1) and (50_2) that the discrete factorized model (49)

$$(E + 0.5\tau A^{(1)})(E + 0.5\tau A^{(2)})y_{\bar{t}}^{n+1} + Ay^n = 0 \tag{53}$$

used in the passage $y^n \to y^{n+1}$ in the standard implementation (50) $\leftrightarrow$ (50_1), (50_2) permits one to express y^{n+1} via y^n alone in *closed form*. This is also true for the factorized model

$$(E + 0.5\tau A^{(2)})(E + 0.5\tau A^{(1)})y_{\bar{t}}^{n+1} + Ay^n = 0. \tag{54}$$

Such discrete models can naturally be refereed to as *solvable in closed form*. Obviously, the parallelization degree for models solvable in closed form is the same as for ordinary difference schemes, where $B = E$ in the canonical representation (48). We especially note the drastic difference of the passage $y^n \to y^{n+1}$ from the corresponding passages in (53) and (54). The operator B_1 is lower triangular, and the operator B_2 is upper triangular; therefore, in this case, the factorization method implements *approximate* factorized representations of the unknown operator B in the form $B = B_1B_2 + O(\tau^2)$ for the model (53) and $B = B_2B_1 + O(\tau^2)$ for the model (54). Let (B) be the

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matrix of the operator B in a Cartesian basis. Then the analogy with similar (exact) representations $(B) = (B_1)(B_2)$ or $(B) = (B_2)(B_1)$ in the classical Gauss elimination method is obvious. Note also that the models (53) and (54) solvable in closed form have conservation laws different from (31). However, in this case, one can estimate the mechanical energy imbalance, which, say, for (53), can be represented as follows [3]:

$$1 - I_h^0/I_h^k \le 0.25\omega^2 = \varepsilon_1. \tag{55}$$

Here $\omega = N_1\tau/h$ is the "Courant number," and ε_1 is the prescribed value of the "imbalance" after the termination of computations at $t_1 = k\tau$. The domain constant N_1 is independent of τ and h. Therefore, one can consider the model (53) as a discrete model with controlled mechanical energy imbalance solvable in closed form.

In accordance with the preceding, we seek efficient implementations of the viscoelastic discrete model (48) in the class of explicitly solvable models, which admit "high"-degree parallelization. As a starting point, we use the additive decomposition

$$\widehat{A} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & 0 \end{pmatrix} = \begin{pmatrix} A_1 & A_{12} \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} A_2 & 0 \\ A_{21} & 0 \end{pmatrix} = \widehat{A}^{(1)} + \widehat{A}^{(2)}, \qquad A_{11} = A_1 + A_2. \tag{56}$$

Next, from (48) and (56), we pass to the factorized model

$$(E + 0.5\tau\widehat{A}^{(1)})(E + 0.5\tau\widehat{A}^{(2)})\widehat{y}_{\bar{t}}^{\,n+1} + \widehat{A}\widehat{y}^{\,n} = 0 \tag{57}$$

for which we use the standard implementation (50): $\widehat{y}^{\,n} \to \widehat{y}^{\,*} \to \widehat{y}_{\bar{t}}^{\,n+1}$ and obtain

$$(E + 0.5\tau\widehat{A}^{(1)})\widehat{y}^{\,*} + \widehat{A}\widehat{y}^{\,n} = 0, \qquad (E + 0.5\tau\widehat{A}^{(2)})\widehat{y}_{\bar{t}}^{\,n+1} = \widehat{y}^{\,*}, \tag{58}$$

or, which is equivalent,

$$\begin{pmatrix} (E + 0.5\tau A_1) & 0.5\tau A_2 \\ 0 & E \end{pmatrix}\begin{pmatrix} \widehat{v}^{\,*} \\ \widehat{\varepsilon}^{\,*} \end{pmatrix} + \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & 0 \end{pmatrix}\begin{pmatrix} \widehat{v}^{\,n} \\ \widehat{\varepsilon}^{\,n} \end{pmatrix} = 0, \tag{58$_1$}$$

$$\begin{pmatrix} (E + 0.5\tau A_2) & 0 \\ 0.5\tau A_{21} & E \end{pmatrix}\begin{pmatrix} \widehat{v}_{\bar{t}}^{\,n+1} \\ \widehat{\varepsilon}_{\bar{t}}^{\,n+1} \end{pmatrix} = \begin{pmatrix} \widehat{v}^{\,*} \\ \widehat{\varepsilon}^{\,*} \end{pmatrix}. \tag{58$_2$}$$

From the formal viewpoint, the method (58$_1$), (58$_2$), as well as the method (50$_1$), (50$_2$), can be treated as an operator alternating-triangular method implementing the corresponding discrete models (32) and (29) with $\alpha = \beta = 0.5$. Moreover, in (58$_1$), (58$_2$), as well as in (50$_1$), (50$_2$), one can use the following order of determining the components of the covector $\widehat{y}$: $\widehat{v}^{\,*} \to \widehat{\varepsilon}^{\,*}$ and $\widehat{v}_{\bar{t}}^{\,n+1} \to \widehat{\varepsilon}_{\bar{t}}^{\,n+1}$. However, unlike (50$_1$), (50$_2$), the structure of the operators

$$(E + 0.5\tau A_m): \ H_h \to H_h, \qquad m = 1, 2,$$

is undefined in (58$_1$), (58$_2$). It is only known that $A_1 + A_2 = A_{11} : \ H_h \to H_h$. Therefore, an economical algorithm for the inversion of the operators $(E+0.5\tau A_m)$ in (58$_1$), (58$_2$) is impossible in the general case.

Lemma 6 [11, 12]. *The Lamé grid operator $L_h(\eta) = R^*\eta R_h = \Lambda_h$ admits the decomposition*

$$\Lambda_h = \Lambda_h^{(1)} + \Lambda_h^{(2)}, \qquad (\Lambda_h^{(1)})^* = \Lambda_h^{(2)}. \tag{59}$$

As was already mentioned, for the adjoint-coordinated approximations (28), the operator R_h (or R_h^*) is defined to within a specific approximation to the operator ∂_j in (8). To be definite,

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we assume that $\partial_j^+ \simeq \partial_j$ for R_h; then for R_h^* one should set $\partial_j^- \simeq \partial_j$. This permits one to represent Λ_h in (59) as follows:

$$\Lambda_h = \begin{pmatrix} \Lambda_{11} & \Lambda_{21}^* & \Lambda_{31}^* \\ \Lambda_{21} & \Lambda_{22} & \Lambda_{32}^* \\ \Lambda_{31} & \Lambda_{32} & \Lambda_{33} \end{pmatrix} = \Lambda_h^* > 0,$$

where

$$\begin{aligned} &\Lambda_{11} = -(\eta_1(\cdot)_{x_1})_{\bar{x}_1} - (\eta_2(\cdot)_{x_2})_{\bar{x}_2} - (\eta_2(\cdot)_{x_3})_{\bar{x}_3}, \\ &\Lambda_{22} = -(\eta_2(\cdot)_{x_1})_{\bar{x}_1} - (\eta_1(\cdot)_{x_2})_{\bar{x}_2} - (\eta_2(\cdot)_{x_3})_{\bar{x}_3}, \\ &\Lambda_{33} = -(\eta_2(\cdot)_{x_1})_{\bar{x}_1} - (\eta_2(\cdot)_{x_2})_{\bar{x}_2} - (\eta_1(\cdot)_{x_3})_{\bar{x}_3}, \qquad -\Lambda_{ij} = -(\eta_2(\cdot)_{x_i})_{\bar{x}_j}, \qquad i > j. \end{aligned} \tag{60}$$

Then it follows from definitions (60) that

$$\Lambda_{ii} = \Lambda_{ii}^{(1)} + \Lambda_{ii}^{(2)}, \qquad (\Lambda_{ii}^{(1)})^* = (\Lambda_{ii}^{(2)}). \tag{61}$$

This permits one to set $A_1 = \Lambda_h^{(1)}$ in (58_1) and $A_2 = \Lambda_h^{(2)}$ in (58_2). We have thereby formally constructed an "economical" passage $\widehat{y}^{\,n} \to \widehat{y}^{\,n+1}$: $(58_1) \to (58_2)$.

Remark 4. As was several times mentioned above, the passage $B \to B_1, \ldots, B_m$ in the canonical representation of the considered viscoelastic model

$$B\widehat{y}_{\bar{t}}^{\,n+1} + \widehat{A}\widehat{y}^{\,n} = 0, \qquad \widehat{A} = \begin{pmatrix} \Lambda_h & A_{12} \\ A_{21} & 0 \end{pmatrix},$$

is always associated with a particular additive decomposition of the operator $\widehat{A}$. We have $m = 2$ in the model (58); however, the order of determination of components of the covector $\widehat{y}$ is fixed in (58_1), (58_2); namely,

$$\widehat{y}^{\,n} = \begin{pmatrix} \widehat{v}^{\,n} \\ \widehat{\varepsilon}^{\,n} \end{pmatrix} \to \begin{pmatrix} \widehat{v}^{\,*} \\ \widehat{\varepsilon}^{\,n} \end{pmatrix} \to \begin{pmatrix} \widehat{v}^{\,*} \\ \widehat{\varepsilon}^{\,*} \end{pmatrix} \to \begin{pmatrix} \widehat{v}^{\,n+1} \\ \widehat{\varepsilon}^{\,*} \end{pmatrix} \to \begin{pmatrix} \widehat{v}^{\,n+1} \\ \widehat{\varepsilon}^{\,n+1} \end{pmatrix} = \widehat{y}^{\,n+1}. \tag{62}$$

At each stage of the passage $\widehat{y}^{\,n} \to \widehat{y}^{\,n+1}$ in (62), the notions "efficiency" and "parallelization degree" have different meanings. Let us clarify this assertion. Let

$$\widehat{A} = \begin{pmatrix} 0 & A_{12} \\ A_{21} & 0 \end{pmatrix} + \begin{pmatrix} \Lambda_h & 0 \\ 0 & 0 \end{pmatrix} = \widehat{A}^{(1)} + \widehat{A}^{(2)}.$$

This decomposition corresponds to the factorized viscoelastic model

$$(E + 0.5\tau\widehat{A}^{(1)})(E + 0.5\tau\widehat{A}^{(2)})\widehat{y}_{\bar{t}}^{\,n+1} + \widehat{A}\widehat{y}^{\,n} = 0,$$

but

$$\begin{aligned} &\widehat{A}^{(1)} = \begin{pmatrix} 0 & A_{12} \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ A_{21} & 0 \end{pmatrix} = \widehat{A}_1^{(1)} + \widehat{A}_2^{(1)}, \\ &\widehat{A}^{(2)} = \begin{pmatrix} \Lambda_h^{(1)} & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} \Lambda_h^{(2)} & 0 \\ 0 & 0 \end{pmatrix} = \widehat{A}_1^{(2)} + \widehat{A}_1^{(2)}, \qquad (\Lambda_h^{(1)})^* = \Lambda_h^{(2)}. \end{aligned}$$

Then the passage $\widehat{y}^{\,n} \to \widehat{y}^{\,n+1}$ in the standard implementation (50) is carried out as follows ([cf. (62)]:

$$\widehat{y}^{\,n} = \begin{pmatrix} \widehat{v}^{\,n} \\ \widehat{\varepsilon}^{\,n} \end{pmatrix} \to \begin{pmatrix} v^* \\ \varepsilon^* \end{pmatrix} \to \begin{pmatrix} v^{n+1} \\ \varepsilon^{n+1} \end{pmatrix} \to \begin{pmatrix} \widehat{v}^{\,*} \\ \varepsilon^{n+1} \end{pmatrix} \to \begin{pmatrix} \widehat{v}^{\,n+1} \\ \varepsilon^{n+1} \end{pmatrix} \to \begin{pmatrix} \widehat{v}^{\,n+1} \\ \widehat{\varepsilon}^{\,n+1} \end{pmatrix} = \widehat{y}^{\,n+1}. \tag{63}$$

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Under the passage $\widehat{y}^{\,n} \to y^{n+1}$ in (63), we deal with a model solvable in closed form. The number of arithmetic operations for the computation of components of the covector y^* or y^{n+1} in (63) at a node $(M_h, (n+1)\tau)$ is independent of the total number of nodes (efficiency). The parallelism degree is the same as in the ordinary explicit scheme. The situation is different under the passage $y^{n+1} \to \widehat{y}^{\,n+1}$. Only the component v of the covector $y = (v, \varepsilon)^{\mathrm{T}}$ changes under this passage, and the passage itself consists of two stages, (i) $v^{n+1} \to \widehat{v}^{\,*}$ and (ii) $\widehat{v}^{\,*} \to \widehat{v}^{\,n+1}$. For each of these stages, *there exist* numberings of nodes of the grid M_h^m, that specify the order of finding $\widehat{v}_h^{\,*}$ or $\widehat{v}_h^{\,n+1}$:

$$(M_h^1)^* \to (M_h^2)^* \to \cdots \to (M_h^{m_1})^*; \qquad (M_h^1)^{n+1} \to (M_h^2)^{n+1} \to \cdots \to (M_h^{m_1})^{n+1}$$

for which the discrete models at these stages become solvable in closed form. In this connection, we have the problem of constructing a numbering strategy for which the paralellization degree is maximal. Obviously, the solution of this problem depends on the computer configuration used in the computations.

In conclusion, note that, for the above-considered implementations of the discrete viscoelastic model (32) with $\alpha = \beta = 0.5$, the basic constants in (37) and [12] permit one to obtain an estimate similar to (55).

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